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Arc length tests for comparing the dynamics between GARCH processes

Ferebee Tunno and Javier Muñoz Ruiz

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We present an arc length test that compares the conditional volatility structures between independent, stationary GARCH processes. A wide variety of simulations are conducted that reveal the power and error of this test to be reasonable and robust with some exceptions. An application involving the daily returns from four major penny cryptocurrencies is presented as well. The years 2020 and 2021 are considered, but since the daily closing prices in 2021 behave very differently from those in 2020, the two years are treated separately.

1. Introduction

The concept of autoregressive conditional heteroscedastic (ARCH) processes was introduced in [Engle 1982]. Specifically, if $\{\epsilon_t\} \sim \mathrm{ARCH}(p)$, then

$$\epsilon_t = \sigma_t Z_t, \quad t \in \mathbb{Z},$$

where $\{Z_t\} \sim \mathrm{IID}(0, 1)$ and

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2,$$

with $\alpha_0, \alpha_p > 0$ and $\alpha_1, \ldots, \alpha_{p-1} \geq 0$. It then follows that $\mathbb{E}(\epsilon_t) = 0$ and

$$\mathrm{Cov}(\epsilon_t^2, \epsilon_{t+h}^2) = \mathrm{Cov}(\sigma_t^2, \sigma_{t+h}^2) \quad \text{for all } h \in \mathbb{Z}.$$

Furthermore, if $\mathcal{E}_t = \sigma(\epsilon_t, \epsilon_{t-1}, \ldots)$, then $\mathrm{Var}(\epsilon_t \mid \mathcal{E}_{t-1}) = \sigma_t^2$. In the special case where $\sum_{i=1}^{p} \alpha_i < 1$, there exists a stationary and causal solution to the ARCH difference equations:

$$\mathrm{Cov}(\epsilon_t, \epsilon_{t+h}) = \begin{cases} \alpha_0 / \bigl(1 - \sum_{i=1}^{p} \alpha_i\bigr), & h = 0, \\ 0, & h \neq 0. \end{cases}$$

See Appendix A for proofs.

MSC2020: 60G10.
Keywords: conditional volatility, arc length, GARCH, cryptocurrency.

The concept of generalized autoregressive conditional heteroscedastic (GARCH) processes was introduced in [Bollerslev 1986]. Specifically, if $\{\epsilon_t\} \sim \text{GARCH}(p, q)$, then

$$\epsilon_t = \sigma_t Z_t, \quad t \in \mathbb{Z},$$

where $\{Z_t\} \sim \text{IID}(0, 1)$ and

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

with $\alpha_0, \alpha_p, \beta_q > 0$ and $\alpha_1, \dots, \alpha_{p-1}, \ \beta_1, \dots, \beta_{q-1} \geq 0$. It then follows that $\mathbb{E}(\epsilon_t) = 0$ and

$$\text{Cov}(\epsilon_t^2, \epsilon_{t+h}^2) = \text{Cov}(\sigma_t^2, \sigma_{t+h}^2) \quad \text{for all } h \in \mathbb{Z}.$$

Furthermore, if $\mathcal{F}_t = \sigma(\epsilon_t, \sigma_t, \epsilon_{t-1}, \sigma_{t-1}, \dots)$, then $\text{Var}(\epsilon_t \,|\, \mathcal{F}_{t-1}) = \sigma_t^2$. In the special case where $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, there exists a stationary and causal solution to the GARCH difference equations:

$$\text{Cov}(\epsilon_t, \epsilon_{t+h}) = \begin{cases} \alpha_0 / \big(1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j\big), & h = 0, \\ 0, & h \neq 0. \end{cases}$$

See Appendix B for proofs.

This article is concerned with comparing the conditional volatility (CV) structures among GARCH processes. Specifically, if $\{\epsilon_{t,A}\}$ and $\{\epsilon_{t,B}\}$ are independent, stationary GARCH processes with $X_t := \epsilon_{t,A}^2$ and $Y_t := \epsilon_{t,B}^2$, then comparing the dynamics between CV series $\{\sigma_{t,A}^2\}$ and $\{\sigma_{t,B}^2\}$ is equivalent to testing

$$\begin{aligned} & H_0 : \text{Cov}(X_t, X_{t+h}) = \text{Cov}(Y_t, Y_{t+h}) \quad \text{for all } h \\ \text{vs.} \quad & \\ & H_1 : \text{Cov}(X_t, X_{t+h}) \neq \text{Cov}(Y_t, Y_{t+h}) \quad \text{for at least one } h. \end{aligned}$$

Since $\{X_t\}$ and $\{Y_t\}$ are also independent and stationary, then the above test will henceforward be reframed as

$$\begin{aligned} & H_0 : \gamma_X(h) = \gamma_Y(h) \quad \text{for all } h \\ \text{vs.} \quad & \\ & H_1 : \gamma_X(h) \neq \gamma_Y(h) \quad \text{for at least one } h. \end{aligned} \tag{1}$$

To understand how (1) will be run, the concept of "arc length" for discrete-valued time series is outlined in the next section. See Appendix C for some relevant remarks about invertibility.

Comparing the dynamics between independent, stationary time series is nothing new. Coates and Diggle [1986] compare the spectral densities of two series to test for autocovariance equality. This method relies upon the fact that two short memory stationary autocovariance functions are equivalent if and only if the spectral densities agree at all frequencies (except on a set of Lebesgue measure zero).

In the time domain, [Lund et al. 2009] examine differences between the sample autocovariances to make equality conclusions. They devise an asymptotic distribution for a quadratic form of these differences and also prove a chi-squared limit law for the test statistic using Bartlett's asymptotic limit formula; see Chapter 7 of [Brockwell and Davis 1991].

Jin and Wang [2016] present an order selection test for the equality of autocovariances with the ability to detect autocorrelation differences beyond a fixed lag, something not accomplished by the otherwise powerful time domain test of [Lund et al. 2009]. Cirkovic and Fisher [2021] extend this time domain test to cover dependent time series and utilize a bootstrapped statistic to automatically select the test order.

2. Arc length

The scatterplot of an arbitrary time series is shown in Figure 1, left. Typically, one "connects the dots" for better presentation as shown in Figure 1, right. The sum of the lengths of these line segments is called the *arc length*. Specifically, if $\{W_t\}$ is a discrete-valued time series observed at times $t = 1, 2, \ldots, n$, then its arc length is

$$U_n^W = \sum_{t=2}^{n} \sqrt{1 + (W_t - W_{t-1})^2} = \sum_{t=2}^{n} S_t^W.$$

The authors of [Tunno et al. 2012] showed that if two independent, stationary ARMA processes with finite fourth moments are observed over the same period, then a significant difference between their arc lengths implies a significant difference between their autocovariance structures. The authors of [Wickramarachchi et al. 2015] proved a Gaussian functional central limit theorem for sample arc lengths requiring only a finite second moment. Tunno and Perry [2022] showed that if two independent, mean-zero signal-plus-noise processes are observed over the same period, then a significant difference between their arc length magnitudes implies a significant difference between their underlying structures.

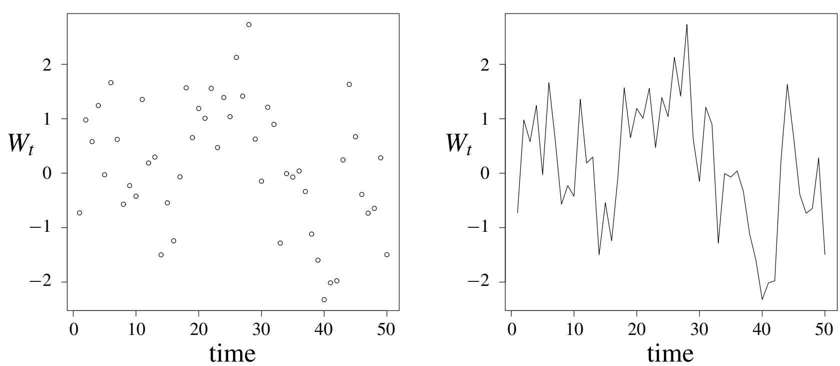

Figure 1. Left: arbitrary time series. Right: connecting the dots.

Arc length test. Let $\{\epsilon_{t,A}\}$ and $\{\epsilon_{t,B}\}$ be independent, stationary GARCH processes with $X_t := \epsilon_{t,A}^2$ and $Y_t := \epsilon_{t,B}^2$. Now suppose that $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$ are samples from $\{X_t\}$ and $\{Y_t\}$, respectively. Since $\{X_t\}$ and $\{Y_t\}$ are stationary, then so are $\{S_t^X\}$ and $\{S_t^Y\}$, in which case

$$\mathrm{Var}(U_n^X) = \mathrm{Var}\left(\sum_{t=2}^{n} S_t^X\right) = (n-1)\gamma_{S^X}(0) + 2\sum_{h=1}^{n-2}(n-1-h)\gamma_{S^X}(h),$$
$$\mathrm{Var}(U_n^Y) = \mathrm{Var}\left(\sum_{t=2}^{n} S_t^Y\right) = (n-1)\gamma_{S^Y}(0) + 2\sum_{h=1}^{n-2}(n-1-h)\gamma_{S^Y}(h).$$

Hence, a normalized test statistic for (1) is given by

$$T = \frac{U_n^X - U_n^Y - \mathbb{E}(U_n^X - U_n^Y)}{\sqrt{\mathrm{Var}(U_n^X - U_n^Y)}} = \frac{U_n^X - U_n^Y - (\mathbb{E}(U_n^X) - \mathbb{E}(U_n^Y))}{\sqrt{\mathrm{Var}(U_n^X) + \mathrm{Var}(U_n^Y)}}.$$

Under the assumption that the null hypothesis implies $\mathbb{E}(U_n^X) = \mathbb{E}(U_n^Y)$, we then have from [Tunno et al. 2012] that

$$T \overset{H_0}{=} \frac{U_n^X - U_n^Y}{\sqrt{\mathrm{Var}(U_n^X) + \mathrm{Var}(U_n^Y)}} \xrightarrow{D} \mathrm{N}(0, 1). \tag{2}$$

Thus, the *arc length test of size* α tells us to reject the null hypothesis if $|T| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the standard normal critical value with area $\alpha/2$ to its right.

3. Simulations

This section compares the type I error and power of arc length for testing (1) at significance level $\alpha = 0.05$. For each figure, series of length $n = 1,000$ were generated while $10,000$ independent simulations were conducted to estimate the error and power values. The Z_t's are i.i.d. standard normal.

In practice, $\gamma_S(h)$ is unknown and so the variances in (2) have been replaced with consistent estimators of the form

$$\widehat{\mathrm{Var}}\left(\sum_{t=2}^{n} S_t\right) = (n-1)\hat{\gamma}_S(0) + 2\sum_{h=1}^{\lfloor\sqrt[3]{n}\rfloor}(n-1-h)\hat{\gamma}_S(h). \tag{3}$$

The estimator

$$\hat{\gamma}_S(h) = \frac{1}{n-1}\sum_{t=2}^{n-h}(S_t - \bar{S})(S_{t+h} - \bar{S}), \quad 0 \le h \le n-2,$$

is nonnegative definite with

$$\bar{S} = \frac{1}{n-1}\sum_{t=2}^{n} S_t.$$

The sum in (3) is truncated at $\lfloor\sqrt[3]{n}\rfloor = 10$ to avoid any bias associated with large lags. The authors of [Berkes et al. 2009] discuss this and other truncation schemes.

k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	5.59	5.16	5.75	5.30	4.79	4.06	3.69	3.25	2.87

Table 1. Percentage type I error with $\alpha_A = \alpha_B = k$.

α_B	$\alpha_A =$ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	5.78	17.81	58.30	90.79	97.89	97.45	94.86	90.71	84.99
0.5	97.98	91.60	62.37	20.52	4.40	18.18	46.85	69.13	73.87
0.9	84.93	84.28	83.27	80.69	74.90	54.67	27.31	8.01	2.64

Table 2. Percentage power with stated values of α_A and α_B.

k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	5.47	5.12	5.50	4.67	4.70	4.32	3.79	3.54

k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	5.62	5.43	5.51	4.64	4.57	4.25	3.37	2.89

Table 3. Percentage type I error with $\alpha_A = \alpha_B = 0.1$ and $\alpha'_A = \alpha'_B = k$ (top), and $\alpha'_A = \alpha'_B = 0.1$ and $\alpha_A = \alpha_B = k$ (bottom).

α'_B	$\alpha'_A =$ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	5.40	31.57	82.15	97.03	97.52	95.68	91.74	85.62
0.45	97.77	89.14	46.77	9.00	7.95	33.30	61.29	72.34
0.8	85.45	84.69	83.27	77.95	62.65	31.98	10.19	3.58

Table 4. Percentage power with $\alpha_A = \alpha_B = 0.1$ and stated values of α'_A and α'_B.

ARCH(1). Let $\{\epsilon_{t,A}\}$ and $\{\epsilon_{t,B}\}$ be independent, stationary ARCH(1) processes, with

$$\sigma^2_{t,A} = 1 + \alpha_A \epsilon^2_{t-1,A} \quad \text{and} \quad \sigma^2_{t,B} = 1 + \alpha_B \epsilon^2_{t-1,B}.$$

Tables 1–2 show type I error and power values for this scenario when testing (1).

ARCH(2). Let $\{\epsilon_{t,A}\}$ and $\{\epsilon_{t,B}\}$ be independent, stationary ARCH(2) processes with

$$\sigma^2_{t,A} = 1 + \alpha_A \epsilon^2_{t-1,A} + \alpha'_A \epsilon^2_{t-2,A} \quad \text{and} \quad \sigma^2_{t,B} = 1 + \alpha_B \epsilon^2_{t-1,B} + \alpha'_B \epsilon^2_{t-2,B}.$$

Tables 3–5 show type I error and power values for this scenario when testing (1).

α_B	$\alpha_A =$ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	5.64	18.59	58.84	89.78	96.82	94.64	90.74	84.74
0.45	95.23	79.93	38.93	8.18	7.52	30.93	57.73	70.07
0.8	85.73	84.12	82.53	77.31	60.12	29.83	9.01	2.84

Table 5. Percentage power with $\alpha'_A = \alpha'_B = 0.1$ and stated values of α_A and α_B.

k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	5.29	5.65	5.25	5.66	5.47	5.76	6.09	9.41	63.8

Table 6. Percentage type I error with $\beta_A = \beta_B = k$.

β_B	$\beta_A =$ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	5.41	41.74	95.82	100	100	100	100	100	99.98
0.5	100	100	99.92	78.05	5.53	91.39	100	100	99.99
0.9	99.97	99.99	99.97	99.98	100	99.98	99.99	99.98	63.62

Table 7. Percentage power with stated values of β_A and β_B.

k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	5.84	4.98	5.46	5.56	5.96	5.19	6.03	9.19

k	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	5.17	5.65	5.80	5.54	5.41	5.54	5.59	5.84

Table 8. Percentage type I error with $\alpha_A = \alpha_B = 0.1$ and $\beta_A = \beta_B = k$ (top), and $\beta_A = \beta_B = 0.1$ and $\alpha_A = \alpha_B = k$ (bottom).

GARCH(0, 1). Let $\{\epsilon_{t,A}\}$ and $\{\epsilon_{t,B}\}$ be independent, stationary GARCH(0, 1) processes with

$$\sigma_{t,A}^2 = 1 + \beta_A \sigma_{t-1,A}^2 \quad \text{and} \quad \sigma_{t,B}^2 = 1 + \beta_B \sigma_{t-1,B}^2.$$

Tables 6–7 show type I error and power values for this scenario when testing (1).

GARCH(1, 1). Let $\{\epsilon_{t,A}\}$ and $\{\epsilon_{t,B}\}$ be independent, stationary GARCH(1, 1) processes with

$$\sigma_{t,A}^2 = 1 + \alpha_A \epsilon_{t-1,A}^2 + \beta_A \sigma_{t-1,A}^2,$$
$$\sigma_{t,B}^2 = 1 + \alpha_B \epsilon_{t-1,B}^2 + \beta_B \sigma_{t-1,B}^2.$$

Tables 8–9 show type I error and power values for this scenario when testing (1).

β_B	$\beta_A =$ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	5.51	41.71	96.01	99.98	100	100	100	100
0.45	100	100	94.79	26.16	30.32	99.67	100	100
0.8	100	100	100	100	100	100	99.99	9.28

α_B	$\alpha_A =$ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	5.75	5.35	5.42	5.55	6.03	5.35	5.63	5.59
0.45	98.11	98.20	98.47	98.34	98.02	98.00	98.52	98.26
0.8	86.13	86.47	86.04	86.45	86.47	86.80	86.74	86.13

Table 9. Percentage power with $\alpha_A = \alpha_B = 0.1$ and stated values of β_A and β_B (top), and with $\beta_A = \beta_B = 0.1$ and stated values of α_A and α_B (bottom).

Conclusions. For an ARCH(1) process, the error hovers around 5% as long as $k < 0.5$. The closer k gets to 1, however, the more the error shrinks. The power increases as the distance between α_A and α_B increases but experiences a dampening effect when at least one of these parameters is close to 1.

For an ARCH(2) process with $\alpha_A = \alpha_B = 0.1$ and $\alpha'_A = \alpha'_B = k$, the error hovers around 5% as long as $k < 0.4$. The closer k gets to 1, however, the more the error shrinks. When $\alpha'_A = \alpha'_B = 0.1$ and $\alpha_A = \alpha_B = k$, the error again hovers around 5% provided that $k < 0.4$. When k approaches 1, the error shrinks even more precipitously than in the previous case.

When $\alpha_A = \alpha_B = 0.1$ and $\alpha'_B \in \{0.1, 0.45, 0.8\}$, the power increases as the distance between α_A and α_B increases but experiences a dampening effect when $\alpha'_A > 0.5$. When $\alpha'_A = \alpha'_B = 0.1$ and $\alpha_B \in \{0.1, 0.45, 0.8\}$, the power increases as the distance between α_A and α_B increases but experiences a dampening effect when $\alpha_A > 0.5$.

For a GARCH(0, 1) process, the error hovers around 5% as long as $k < 0.7$. The closer k gets to 1, however, the more the error blows up. The power values increase as the distance between α_A and α_B increases but are almost certainly inflated when $0.6 < \beta_A < 0.9$. When $\beta_A = 0.9$, another dampening effect occurs, especially when $\beta_B = 0.9$.

For a GARCH(1, 1) process with $\alpha_A = \alpha_B = 0.1$ and $\beta_A = \beta_B = k$, the error hovers around 5% as long as $k < 0.7$. As k gets closer to 1, however, the error begins to rise. When $\beta_A = \beta_B = 0.1$ and $\alpha_A = \alpha_B = k$, the error hovers around 5% for all $k \in \{0.1, 0.2, \ldots, 0.8\}$.

When $\alpha_A = \alpha_B = 0.1$ and $\beta_B \in \{0.1, 0.45, 0.8\}$, the power increases as the distance between β_A and β_B increases, but these values are most likely inflated when $\beta_A > 0.6$. When $\beta_A = \beta_B = 0.1$, the power values behave poorly. Specifically, for all $\alpha_A \in \{0.1, 0.2, \ldots, 0.8\}$, the power hovers around 5% for $\alpha_B = 0.1$, around 98% for $\alpha_B = 0.45$, and around 86% for $\alpha_B = 0.8$.

4. Applications

According to Wikipedia, a *cryptocurrency* is "a digital currency designed to work as a medium of exchange through a computer network that is not reliant on any central authority, such as a government or bank, to uphold or maintain it. It is a decentralized system for verifying that the parties to a transaction have the money they claim to have, eliminating the need for traditional intermediaries, such as banks, when funds are being transferred between two entities". Unlike traditional currencies, cryptocurrencies are traded 365 days a year.

There has been a growing body of literature in recent years that explores the modeling of daily returns from cryptocurrencies. The authors of [Chu et al. 2017] were among the first to describe the modeling of returns from several leading cryptocurrencies using various GARCH-type models. Gkillias and Katsiampa [2018] employed extreme value theory to study the tail behavior of returns in an effort to better understand why cryptocurrencies are more volatile than fiat currencies. Caporale and Zekokh [2019] revealed that standard GARCH models tend to yield incorrect value-at-risk (VaR) predictions and put forward remedies that better accommodate regime changes. The authors of [Cerqueti et al. 2020] revealed that certain skewed GARCH volatility models outperform more traditional Gaussian models when forecasting the market capitalization of certain cryptocurrencies.

Figure 2 shows the daily adjusted closing prices (in USD) for Dogecoin (DOGE), Cardano (ADA), Tron (TRX), and Hex (HEX) from January 1, 2020 to December 31, 2021 ($n = 731$). Changepoints seemingly occur around the turn of the year, making 2021 much more volatile than 2020. Part of the explanation for this shift is that the shutting down of the U.S. economy in 2020 due to COVID-19 stirred up fears of inflationary pressure on the U.S. dollar, thus making cryptocurrencies more attractive to certain investors.

The four aforementioned cryptocurrencies are examples of what are called *penny cryptocurrencies*. Like penny stocks, penny cryptocurrencies trade for less than one USD per share. This section is concerned with using arc length to compare the CV structures of the daily returns among these four penny cryptocurrencies, treating 2020 and 2021 separately.

2020. Figure 3 shows the daily returns for our four chosen cryptocurrencies from January 1, 2020 to December 31, 2020 ($n = 366$). Table 10 shows the test statistic values when testing (1) pairwise among these returns. If $\alpha = 0.05$, then the Bonferroni-corrected (threshold) critical value is $z_{(\alpha/6)/2} = z_{.0041} \approx 2.636$, in which case we fail to reject H_0 for all six tests. If $\alpha = 0.10$, then the Bonferroni-corrected (threshold) critical value is $z_{(\alpha/6)/2} = z_{0.0083} \approx 2.394$, in which case we reject H_0 for the tests involving Dogecoin vs. Cardano, Cardano vs. Tron, and Cardano vs. Hex.

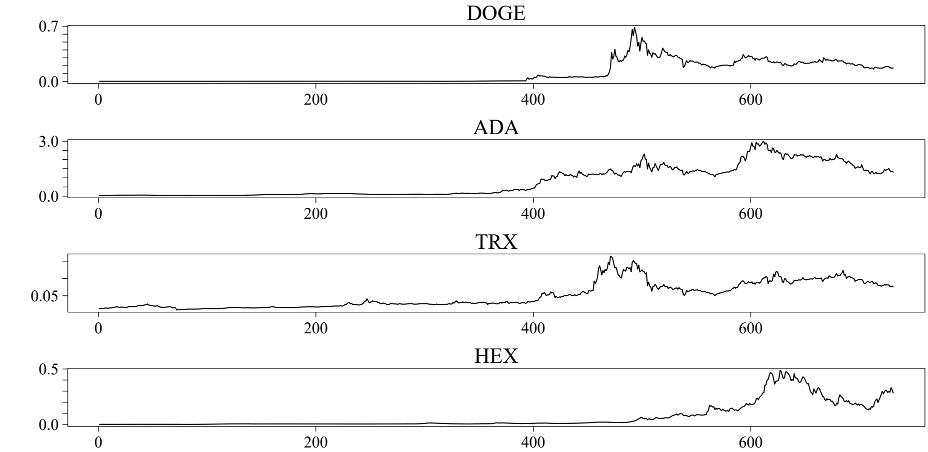

Figure 2. Daily adjusted closing prices (in USD) for Dogecoin (DOGE), Cardano (ADA), Tron (TRX), and Hex (HEX) from January 1, 2020 to December 31, 2021 ($n = 731$).

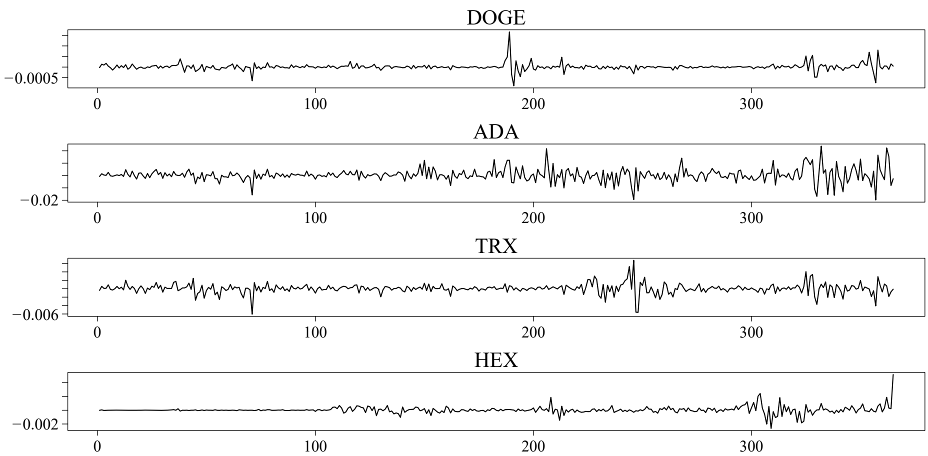

Figure 3. Daily returns for Dogecoin (DOGE), Cardano (ADA), Tron (TRX), and Hex (HEX) from January 1, 2020 to December 31, 2020 ($n = 366$).

	DOGE	ADA	TRX	HEX
DOGE	–	−2.564	−1.898	−1.311
ADA	2.564	–	2.556	2.563
TRX	1.898	−2.556	–	1.676
HEX	1.311	−2.563	−1.676	–

Table 10. Test statistic values when testing (1) pairwise among the 2020 daily returns for Dogecoin (DOGE), Cardano (ADA), Tron (TRX), and Hex (HEX).

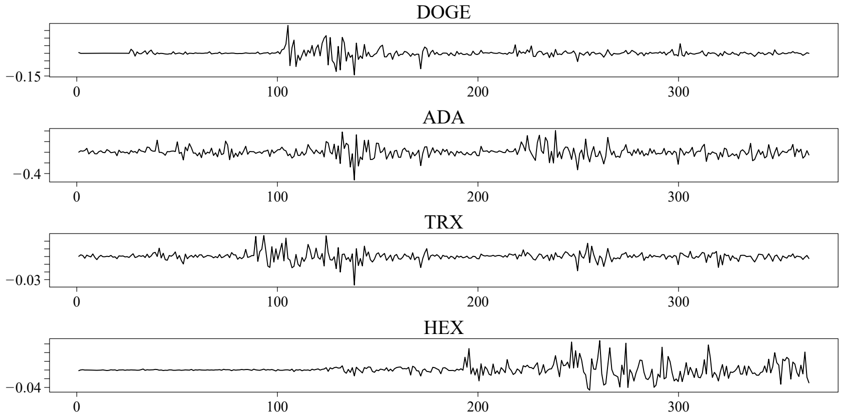

Figure 4. Daily returns for Dogecoin (DOGE), Cardano (ADA), Tron (TRX), and Hex (HEX) from January 1, 2021 to December 31, 2021 ($n = 365$).

	DOGE	ADA	TRX	HEX
DOGE	–	−2.082	1.775	1.679
ADA	2.082	–	2.105	2.104
TRX	−1.775	−2.105	–	−2.464
HEX	−1.679	−2.104	2.464	–

Table 11. Test statistic values when testing (1) pairwise among the 2021 daily returns for Dogecoin (DOGE), Cardano (ADA), Tron (TRX), and Hex (HEX).

2021. Figure 4 shows the daily returns for our four chosen cryptocurrencies from January 1, 2021 to December 31, 2021 ($n = 365$). Table 11 shows the test statistic values when testing (1) pairwise among these returns. If $\alpha = 0.05$, then the Bonferroni-corrected (threshold) critical value is

$$z_{(\alpha/6)/2} = z_{0.0041} \approx 2.636,$$

in which case we fail to reject H_0 for all six tests. If $\alpha = 0.10$, then the Bonferroni-corrected (threshold) critical value is

$$z_{(\alpha/6)/2} = z_{0.0083} \approx 2.394,$$

in which case we reject H_0 for the test involving Tron vs. Hex.

Appendix A: ARCH proofs

If $\{\epsilon_t\} \sim \text{ARCH}(p)$, then

$$\mathbb{E}(\epsilon_t) = \mathbb{E}(\sigma_t Z_t) = \mathbb{E}(\sigma_t)\mathbb{E}(Z_t) = 0$$

and

$$\begin{aligned}
\operatorname{Cov}(\epsilon_t^2, \epsilon_{t+h}^2) &= \mathbb{E}(\epsilon_t^2\epsilon_{t+h}^2) - \mathbb{E}(\epsilon_t^2)\mathbb{E}(\epsilon_{t+h}^2) \\
&= \mathbb{E}(\sigma_t^2 Z_t^2 \sigma_{t+h}^2 Z_{t+h}^2) - \mathbb{E}(\sigma_t^2 Z_t^2)\mathbb{E}(\sigma_{t+h}^2 Z_{t+h}^2) \\
&= \mathbb{E}(\sigma_t^2\sigma_{t+h}^2)\mathbb{E}(Z_t^2)\mathbb{E}(Z_{t+h}^2) - \mathbb{E}(\sigma_t^2)\mathbb{E}(\sigma_{t+h}^2)\mathbb{E}(Z_t^2)\mathbb{E}(Z_{t+h}^2) \\
&= \mathbb{E}(\sigma_t^2\sigma_{t+h}^2) - \mathbb{E}(\sigma_t^2)\mathbb{E}(\sigma_{t+h}^2) \\
&= \operatorname{Cov}(\sigma_t^2, \sigma_{t+h}^2).
\end{aligned}$$

If $\mathcal{E}_t = \sigma(\epsilon_t, \epsilon_{t-1}, \dots)$, then

$$\begin{aligned}
\operatorname{Var}(\epsilon_t \,|\, \mathcal{E}_{t-1}) &= \mathbb{E}(\epsilon_t^2 \,|\, \mathcal{E}_{t-1}) = \mathbb{E}(\sigma_t^2 Z_t^2 \,|\, \mathcal{E}_{t-1}) \\
&= \sigma_t^2 \mathbb{E}(Z_t^2 \,|\, \mathcal{E}_{t-1}) = \sigma_t^2 \mathbb{E}(Z_t^2) = \sigma_t^2.
\end{aligned}$$

If $h \neq 0$, then

$$\begin{aligned}
\operatorname{Cov}(\epsilon_t, \epsilon_{t+h}) &= \mathbb{E}(\epsilon_t\epsilon_{t+h}) - \mathbb{E}(\epsilon_t)\mathbb{E}(\epsilon_{t+h}) \\
&= \mathbb{E}(\epsilon_t\epsilon_{t+h}) = \mathbb{E}(\sigma_t Z_t \sigma_{t+h} Z_{t+h}) \\
&= \mathbb{E}(\sigma_t\sigma_{t+h})\mathbb{E}(Z_t)\mathbb{E}(Z_{t+h}) = 0.
\end{aligned}$$

If $h = 0$, then

$$\begin{aligned}
\operatorname{Cov}(\epsilon_t, \epsilon_{t+h}) &= \operatorname{Var}(\epsilon_t) = \mathbb{E}(\epsilon_t^2) = \mathbb{E}(\sigma_t^2) \\
&= \mathbb{E}\left(\alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2\right) = \alpha_0 + \sum_{i=1}^{p} \alpha_i \mathbb{E}(\epsilon_{t-i}^2).
\end{aligned}$$

A weakly stationary solution (i.e., $\mathbb{E}(\epsilon_s^2) = \mathbb{E}(\epsilon_t^2)$ for all $s, t \in \mathbb{Z}$) for this equation is given by

$$\operatorname{Var}(\epsilon_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i},$$

provided that $\sum_{i=1}^{p} \alpha_i < 1$.

Appendix B: GARCH proofs

If $\{\epsilon_t\} \sim \mathrm{GARCH}(p, q)$, then

$$\mathbb{E}(\epsilon_t) = 0 \quad \text{and} \quad \operatorname{Cov}(\epsilon_t^2, \epsilon_{t+h}^2) = \operatorname{Cov}(\sigma_t^2 \sigma_{t+h}^2)$$

for the same reasons as shown in Appendix A. If $\mathcal{F}_t = \sigma(\epsilon_t, \sigma_t, \epsilon_{t-1}, \sigma_{t-1}, \dots)$, then

$$\begin{aligned}
\operatorname{Var}(\epsilon_t \,|\, \mathcal{F}_{t-1}) &= \mathbb{E}(\epsilon_t^2 \,|\, \mathcal{F}_{t-1}) = \mathbb{E}(\sigma_t^2 Z_t^2 \,|\, \mathcal{F}_{t-1}) \\
&= \sigma_t^2 \mathbb{E}(Z_t^2 \,|\, \mathcal{F}_{t-1}) = \sigma_t^2 \mathbb{E}(Z_t^2) = \sigma_t^2.
\end{aligned}$$

If $h \neq 0$, then $\mathrm{Cov}(\epsilon_t, \epsilon_{t+h}) = 0$ for the same reasons as shown in Appendix A. If $h = 0$, then

$$\begin{aligned}
\mathrm{Cov}(\epsilon_t, \epsilon_{t+h}) &= \mathrm{Var}(\epsilon_t) = \mathbb{E}(\epsilon_t^2) = \mathbb{E}(\sigma_t^2) \\
&= \mathbb{E}\left(\alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2\right) \\
&= \alpha_0 + \sum_{i=1}^{p} \alpha_i \mathbb{E}(\epsilon_{t-i}^2) + \sum_{j=1}^{q} \beta_j \mathbb{E}(\sigma_{t-j}^2) \\
&= \alpha_0 + \sum_{i=1}^{p} \alpha_i \mathbb{E}(\epsilon_{t-i}^2) + \sum_{j=1}^{q} \beta_j \mathbb{E}(\epsilon_{t-j}^2).
\end{aligned}$$

A weakly stationary solution (i.e., $\mathbb{E}(\epsilon_s^2) = \mathbb{E}(\epsilon_t^2)$ for all $s, t \in \mathbb{Z}$) for this equation is given by

$$\mathrm{Var}(\epsilon_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j},$$

provided that $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.

Appendix C: Invertibility

Recall that an autoregressive moving average (ARMA) process $\{A_t\}$ is invertible if there exist absolutely summable values $\pi_0, \pi_1, \pi_2, \ldots$ such that

$$\text{white noise at time } t = \sum_{j=0}^{\infty} \pi_j A_{t-j}.$$

The autocovariance function (ACVF) for an invertible ARMA process uniquely identifies the model parameters.

For example, if $\{A_t\} \sim \mathrm{MA}(1)$, then

$$A_t = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

with ACVF given by

$$\gamma_A(h) = \begin{cases} (\theta^2 + 1)\sigma^2, & h = 0, \\ \theta\sigma^2, & h = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $\theta = 2$ with $\sigma^2 = 0.2$ or if $\theta = 0.5$ with $\sigma^2 = 0.8$, then

$$\gamma_A(h) = \begin{cases} 1, & h = 0, \\ 0.4, & h = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

With the invertibility requirement of $|\theta| < 1$, however, the scenario with $\theta = 2$ gets tossed out, in which case the ACVF now uniquely identifies the MA(1) parameters.

It can be shown that if $\{\epsilon_t\} \sim \mathrm{GARCH}(p, q)$, then $\{\epsilon_t^2\} \sim \mathrm{ARMA}(m, q)$, where $m = \max(p, q)$. Thus, for this paper, we tacitly assume ARMA invertibility so that the null hypothesis in (1) is unambiguous.

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